数理解析研究所講究録
1080 巻 1999 年 149-155

作用素不等式二題
Exponential operator inequalities

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Section 1.

Let X be a unital Banach algebra over $\mathbf{R}$ or $\mathbf{C}$, that is, a complete normed algebra with a unit 1 such that $||1|| = 1$.
The aim of this note is, roughly speaking, to show that if $f : [0.\infty) \to X$ satisfies $f(0) = 1,\ f'(0) = a$, then $f(\frac{t}{n})^n$ converges to e^{ta} as $n \to \infty$, where

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

by definition.
If $X = \mathbf{R}$, this assertion clearly follows from the L'hospital theorem. Since a set of all bounded operators on a Banach space is a unital Banach algebra, for a bounded operator A, e^A is defined as above. In this case for bounded operators A, B the Lie product formula:

$$\exp(A+B) = (n) \lim_{n\to\infty} \{\exp(\frac{A}{n}) \exp(\frac{B}{n})\}^n$$

is well-known, where (n) means that the limit is in the sense of the (operator) norm topology. This implies that the above assertion holds for $f(t) = \exp(tA)\exp(tB)$ as well. The above definition e^x is not useful for unbounded operator. However it is well-known that if A is a generator of (C_0) contractive semigroup, then

$$e^{tA} = (s) \lim_{n\to\infty} (1 - \frac{t}{n}A)^{-n} \quad \text{for} \quad t > 0,$$

where (s) means that the limit is in the sense of strong topology. The Lie product formula was extended to the case of unbounded operators on a Banach space in [2][4].

Chernoff [1] showed a product formula in a more general form as follows :

Let $f(t)$ be a strongly continuous function from $[0,\infty)$ to the linear contractions on a Banach space. Suppose that $f(0)=1$ and the strong derivative $f'(0)$ has a closure A which is a generator of a (C_0) contractive semigroup. Then $f(t/n)^n$ strongly converges to e^{tA}.

In the proof of this theorem the condition $||f(t)|| \leq 1$ plays an important role, so it is not easy to relax it. However we encounter many cases where $f(t)$ is not a contraction and the derivative A is bounded : in this case

$$\frac{f(t)}{||f(t)||}$$

is a contraction, but may not be differentiable at $t=0$; so we can not use the Chernoff's theorem. Therefore we need to make a new product formula for bounded operators. See [3] for product formulas.

Theorem 1. *Let X be a unital Banach algebra, and let $f(t)$ be a function from an interval $0 \leq t < \zeta$ to X. If $f(0)=1$ and $f(t)$ has a norm right derivative a at $t=0$, then*

$$||f(\frac{t}{n})^n - \exp(ta)|| \to 0 \ \ (n \to \infty) \ \ for \ \ 0 \leq t < \infty.$$

Proof. For every $t : 0 \leq t < \infty$, $f(\frac{t}{n})$ is defined for sufficiently large n, so we may assume f is defined on $[0,\infty)$. We claim that
there is $r>0$ such that $||f(t)||^{\frac{1}{t}}$ is bounded on $(0,r)$.
To see this we may show that $\frac{1}{t}\log||f(t)||$ is bounded above on $0<t<r$.
Since

$$||\frac{f(t)-1}{t} - a|| \to 0 \quad (t \to +0),$$

$\frac{1}{t}(||f(t)||-1)$ is bounded, and $||f(t)|| \to 1$ $(t \to +0)$. Thus

$$\frac{\log||f(t)||}{t} = \begin{cases} \frac{\log||f(t)||-\log 1}{||f(t)||-1}\,\frac{||f(t)||-1}{t} & (||f(t)|| \neq 1) \\ 0 & (||f(t)||=1) \end{cases}$$

is bounded on some interval $(0,r)$.
Now take an arbitrary $t : 0 < t < \infty$, and fix it. By the claim above, we can see that $\{||f(\frac{t}{n})||^n\}_n$ is bounded. Thus there is $M>0$ such that

$$e^{t||a||} \leq M, \ ||f(\frac{t}{n})||^n \leq M \ for \ every \ n.$$

From

$$f(\frac{t}{n})^n - e^{ta} = \sum_{m=0}^{n-1} f(\frac{t}{n})^m \{f(\frac{t}{n}) - e^{\frac{t}{n}a}\}(e^{\frac{t}{n}a})^{n-1-m},$$

it follows that

$$||f(\frac{t}{n})^n - e^{ta}|| \leq ||f(\frac{t}{n}) - e^{\frac{t}{n}a}|| \sum_{m=0}^{n-1} M^{\frac{m}{n}} (e^{\frac{t}{n}||a||})^{n-1-m}$$

$$= n||f(\frac{t}{n}) - e^{\frac{t}{n}a}|| \cdot \frac{M - e^{t||a||}}{n(M^{\frac{1}{n}} - e^{\frac{t}{n}||a||})}.$$

Since

$$n(M^{\frac{1}{n}} - e^{\frac{t}{n}||a||}) \to \log M - t||a||$$

and

$$n||f(\frac{t}{n}) - e^{\frac{t}{n}a}|| \leq t||\frac{n}{t}\{f(\frac{t}{n}) - 1\} - a|| + t||\frac{n}{t}(-e^{\frac{t}{n}a} + 1) + a|| \to 0 \ \ (n \to \infty),$$

we get

$$||f(\frac{t}{n})^n - e^{ta}|| \to 0 \ \ (n \to \infty).$$

This concludes the proof. □

Corollary 1. *For* $a_i \in X \ \ (i = 1, \cdots, m)$

$$||\{(1 + \frac{a_1}{n}) \cdots (1 + \frac{a_m}{n})\}^n - \exp(a_1 + \cdots + a_m)|| \to 0,$$

$$||(e^{\frac{a_1}{n}} \cdots e^{\frac{a_m}{n}})^n - \exp(a_1 + \cdots + a_m)|| \to 0.$$

Proof. By setting $f(t) = (1 + ta_1) \cdots (1 + ta_m)$ or $f(t) = e^{ta_1} \cdots e^{ta_m}$, these follows from the theorem. □

Let $\phi(z)$ be a holomorphic function in a neighborhood $|z - 1| < \delta$. Then for $a \in X$: $||a - 1|| < \delta$, $\phi(a)$ is defined by

$$\phi(a) = \sum_{n=o}^{\infty} \frac{\phi^{(n)}(1)}{n!}(a - 1)^n,$$

which converges in the norm. Thus for $f(t)$ with the property set out in the theorem $\phi(f(t))$ is well-defined for sufficiently small t. Since $\phi(f(0)) = \phi(1)$ and the right norm derivative of $\phi(f(t))$ at $t = 0$ is $\phi'(1)f'(0)$, we have

Corollary 2. *If $\phi(z)$ is a scalar valued holomorphic function in a neighborhood of $z = 1$, with $\phi(1) = 1$, then for $f(t)$ which has the property set out in the theorem,*

$$||\phi(f(\frac{t}{n}))^n - \exp(t\phi'(1)a)|| \to 0 \quad (n \to \infty) \quad for \quad 0 \leq t < \infty.$$

In particular, we have

Corollary 3.

$$||\{(1+\frac{a_1}{n})^{\lambda_1} \cdots (1+\frac{a_m}{n})^{\lambda_m}\}^n - \exp(\lambda_1 a_1 + \cdots + \lambda_m a_m)|| \to 0 \quad for \quad \lambda_i \in \mathbf{R}.$$

In the proof of Theorem 1 that the domain of f is the right half real line is not essential. We can get the same result as above even if the domain of f is a half line with end point 0 in $\mathbf{C}$. More generally we show

Theorem 4. *Let X be a unital Banach algebra, set $D = \{z \in \mathbf{C} : \ \alpha \leq \arg z \leq \beta, \ 0 \leq \alpha \leq 2\pi\}$. If a function $f : D \to X$ satisfies $f(0) = 1$ and $f'(0) = a$, that is,*

$$||\frac{f(z) - f(0)}{z} - a|| \to 0 \quad (z \in D, z \to 0),$$

then for every $z \in D$, $||f(\frac{z}{n})^n - \exp za|| \to 0 \quad (n \to \infty)$.

Proof. In the same way as the proof of Theorem 1 one can easily show that $||f(z)||^{\frac{1}{|z|}}$ is bounded on a neighborhood of $0 \in D$, and that , for fixed $z \in D$,

$$||f(\frac{z}{n})^n - e^{za}|| \leq ||f(\frac{z}{n}) - e^{\frac{z}{n}a}|| \sum_{m=0}^{n-1} M^{\frac{m}{n}} (e^{\frac{|z|}{n}||a||})^{h-1-m},$$

from which the theorem follows. □

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Section 2.

Let A and B be bounded selfadjoint operators on a Hilbert space. The following celebrated inequality was found by Furuta in [4] and simply proved in [5].

$$A \geq B \geq 0 \text{ implies } A^{(p+r)/q} \geq (A^{r/2} B^p A^{r/2})^{1/q} \tag{1}$$

for $p \geq 0, q \geq 1, r \geq 0$ such that $(1+r)q \geq p+r$.

Ando [1] showed the following theorem in the case of $s = p = r$ with a splended idea. Then Fujii, Furuta, Kamai [2], by making use of Ando's result, proved that $A \geq B$ implies (2).

Theorem A. *$A \geq B$ implies that for $p \geq 0,\ r \geq s \geq 0$*

$$e^{sA} \geq (e^{\frac{r}{2}A} e^{pB} e^{\frac{r}{2}A})^{\frac{s}{r+p}}. \tag{2}$$

In [1] Ando also showed the converse :

Theorem B. *If*

$$e^{tA} \geq (e^{\frac{t}{2}A} e^{tpB} e^{\frac{t}{2}A})^{\frac{t}{r+p}} \quad for \quad every \quad t > 0,$$

then $A \geq B$.

The aim of this note is to give a new way to get exponential inequalities from operator inequalities like (1), and to extend Theorems A, B.
We start with a quite simple proof of Theorem A. This technique seems to be very effective

to study operator inequality.

Another proof of Theorem A. For sufficiently large n we have $1+\frac{A}{n} \geq 1+\frac{B}{n} \geq 0$. By substituting np and nr to p and r of (1), respectively, we get,

$$(1+\frac{A}{n})^{\frac{n(p+r)}{q}} \geq \{(1+\frac{A}{n})^{n\frac{r}{2}}(1+\frac{B}{n})^{np}(1+\frac{A}{n})^{n\frac{r}{2}}\}^{1/q}, \; for\; rq \geq p+r.$$

Since for selfadjoint operator X, $(1+\frac{X}{n})^n$ converges to e^X in the operator norm as $n \to \infty$, we gain (2) by setting $s = \frac{p+r}{q}$. □

We slightly extend Theorem A by using itself.

Proposition 1. *$A \geq B$ implies*

$$e^{sA} \geq \{e^{\frac{r}{2}A}e^{(qA+pB)}e^{\frac{r}{2}A}\}^{\frac{s}{(p+q+r)}} \tag{3}$$

for p,q,r,s with $0 \leq s \leq r$, $\quad 0 \leq p, p+q$, and $0 < p+q+r$.

Proof. If $p+q=0$, then $e^{(qA+pB)}$ is contractive, so that the above inequality follows. Therefore we assume that $p+q>0$. Since

$$\frac{qA+pB}{q+p} \leq A,$$

by using (2), we gain (3). □

Now we extend Theorem B:

Theorem 2. *If there are p,q,r,s with $p>0, p+q \geq 0, r \geq s > 0$ such that*

$$e^{stA} \geq \{e^{\frac{rt}{2}A}e^{t(qA+pB)}e^{\frac{rt}{2}A}\}^{\frac{s}{(p+q+r)}}$$

for every $t>0$, then $A \geq B$.

Proof. If $p+q+r=s$, then the above inequality implies that $e^{t(qA+pB)}$ is contractive because of $p+q=0$. Hence $A \geq B$. Suppose $p+q+r>s$. Set

$$f(t) = e^{\frac{-rt}{2}A}e^{-t(qA+pB)}e^{\frac{-rt}{2}A}, \quad g(t) = e^{-stA}.$$

Then we get

$$(f(t)^{\frac{s}{(p+q+r)}}x,x) \geq (g(t)x,x) \quad (||x||=1, \quad t>0),$$

from which it follows that

$$(f(t)x,x)^{\frac{s}{(p+q+r)}} \geq (g(t)x,x) \quad (t>0)$$

because of Jensen's inequality. Since the values of both sides of the inequality above at $t=0$ are 1, the right derivative of the left hand side at $t=0$ is greater than or equal to the one of the right hand side. Since the norm derivative of e^{tT} at $t=0$ is generally T, we have

$$\frac{s}{(p+q+r)}((-\frac{r}{2}A-(qA+pB)-\frac{r}{2}A)x,x) \geq (-sAx,x).$$

Hence we gain $A \geq B$. □

We end this note with referring to an exponential inequality which appeared in [3]:

If $A-B \geq \delta > 0$, *then* $e^{tA}-e^{tB} \geq \delta/2 > 0$ *for some* $t>0$.

This seems to be useful, so that we give a more generalized result, which we can see by a simple calculation.

Let $f(t), g(t)$ *be selfadjoint operator valued functions defined in a neighborhood of* $t=0$. *If* $f(0)=g(0)$ *and* $f'(0)-g'(0) \geq \delta > 0$, *where the derivative is in the sense of norm, then* $f(t)-g(t) \geq \delta/2$ *for* t *in a neighborhood of* 0.

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